AN ELASTIC-PLASTIC PROBLEM UNDER CONDITIONS OF ANTIPLANAR DEFORMATION

(UPRUGO-PLASTICHESKAIA ZADACHA V USLOVIIAKH ANTIPLOSKOI DEFORMATSII)

PMM Vol.26, No.4, 1962, pp. 697-708

G.P. CHEREPANOV
(Moscow)

(Received June 12, 1962)

By antiplanar deformation is meant the state of stress in an infinitely long cylinder subjected to the action of loading which is applied in the direction of the generators and which is constant along them. The elastic-plastic problem under the condition of antiplanar deformation has already been considered in the works of Trefftz [1], Hult and McClintock [2], Neuber [3]. Paper [1] gives an exact solution of the elastic-plastic problem on the antiplanar deformation of an angle section with right-angle opening, and also of the analogous problem for a region exterior to a circular hole. The elastic-plastic problem for a half-plane with a sharp notch has been solved for small values of the loading parameter in [2]. Neuber [3] considered a strip with two symmetric sharp notches, and, moreover, for an arbitrary single-valued relation between the stresses and the strains, the solution of the problem was reduced to a system of two ordinary differential equations and for a specially selected law the solution was obtained in closed form.

Below, a treatment will be given of the solution in quadratures of the static elastic-plastic problem for the exterior of an arbitrary contour wholly enclosed by the plastic zone and loaded arbitrarily (Section 2); an exact solution of the problem for the exterior of a contour consisting of segments of straight and curved lines in the case when the straight sections are free of stresses and the parts of the curved arcs, which are arbitrarily loaded, are wholly contained in the plastic zone (Section 4).

The solutions of the problems of Section 4 are based mainly on the solution of a certain nonlinear boundary value problem (Section 3). Throughout this article the Prandtl diagram has been taken as the relation between the stresses and the strains.

1. General relations. The fields of displacements and stresses in the considered body are such that

$$u = v = 0, \quad w = w(x, y), \quad \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$
$$\tau_{xz} = \tau_{xz}(x, y), \quad \tau_{yz} = \tau_{yz}(x, y) \tag{1.1}$$

Here u, v, w are the components of the displacement vector; σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz} are the components of the stress tensor; x, y, z are Cartesian coordinates (the z-axis is parallel to the generators). In the plastic region we have the relations [4, 5]

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \qquad \text{(equilibrium equation)} \tag{1.2}$$

$$\tau_{xz}^2 + \tau_{yz}^2 = k^2 \qquad \text{(yield condition)} \tag{1.3}$$

$$\tau_{yz} \frac{\partial w}{\partial x} - \tau_{xz} \frac{\partial w}{\partial y} = 0 \qquad \text{(Hencky condition)} \tag{1.4}$$

Here $k = \tau_s$ according to the Huber-von Mises condition, and $k = 2\tau_s/\sqrt{3}$ according to the Tresca-Saint-Venant condition, and τ_s is the yield value in pure shear. The stresses can be determined independently of the boundary of the plastic region. We represent the stresses in the form [5]

$$\tau_{xz} = k \cos \theta, \quad \tau_{yz} = k \sin \theta \tag{1.5}$$

Here the function $\theta(x, y)$ satisfies the condition

$$-\sin \theta \frac{\partial \theta}{\partial x} + \cos \theta \frac{\partial \theta}{\partial y} = 0 \tag{1.6}$$

The characteristics of Equation (1.6) are the family of straight lines $y = -x \cot \theta + C$, $\theta = \text{const}$, which coincide with the slip lines and are orthogonal to the vector $\tau = \tau_{xz} + i\tau_{yz}$ at every point. From the Hencky relation (1.4) it follows that $w = \text{const}$ along the slip lines. Thus the stress field in the plastic region is completely determined by the form of the boundary of the plastic region and by the boundary loading. On the boundary between the elastic and plastic regions we allow no discontinuity in the stresses or the displacement.

In the elastic region the stresses and the displacement can be determined [1] by means of one analytic function of a complex variable (μ is the shear modulus)

$$w = \operatorname{Re} f(z), \qquad \tau = \tau_{xz} + i\tau_{yz} = \mu \overline{f'(z)} \qquad (z = x + iy) \tag{1.7}$$

2. The elastic-plastic problem for an arbitrary hole in an infinite plane in the case when the plastic region completely surrounds the hole. Let the sectionally smooth contour C of the hole be representable in the complex variable z by means of the parametric equations $x = \xi(t)$, $y = \eta(t)$, where $\xi(t)$ and $\eta(t)$ are periodic, single valued functions with the same period T, having sectionally continuous derivatives (Fig. 1a). Applied to the contour there is a load $\tau_{zn} = k\tau(t)$, where $\tau(t)$ is a sectionally continuous function and $|\tau(t)| \leqslant 1$. The contour L of the plastic region contains all of the hole C. By Formula (1.5) on the basis of the boundary data we find that the stresses in the plastic region are

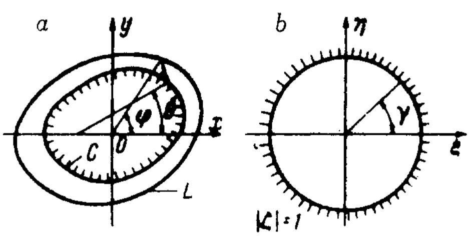

Fig. 1. $|\zeta| = 1$

$$\tau = ke^{i\theta}$$

along the line

$$y - \eta(t) = -(x - \xi(t)) \cot\theta$$
$$\theta = \alpha(t) - \beta(t) \qquad (2.1)$$

Here for simplicity the following notation has been introduced

$$\tau(t) = \cos\alpha(t), \qquad \xi'(t) = \sin\beta(t)\sqrt{\xi'^2(t) + \eta'^2(t)} \qquad (2.2)$$
$$(0 < \alpha(t) < \pi, \quad 0 < \beta(t) < 2\pi)$$

The function $\theta = \theta(t)$, given by the relation (2.1), is sectionally continuous. We require that the inverse function $t = t(\theta)$ should be single-valued.

We assume that every slip line intersects the contour L at one point, and, conversely, that from each point of the contour L it is possible to construct only one slip line proceeding from the boundary C of the body.

It is easy to derive the relation connecting the function θ with the coordinates of the point of the contour of the plastic region $z = |z|e^{i\varphi}$, in which the stress can be determined by means of Formula (2.1) (Fig. 1a)

$$|z|\cos(\varphi - \theta) = \eta(t)\sin\theta + \xi(t)\cos\theta \qquad (2.3)$$

On the contour L of the boundary between the elastic and plastic regions we have, by virtue of Formulas (2.1) and (1.2), the condition for the continuity of the stresses

$$\mu f'(z) = ke^{-i\theta} \tag{2.4}$$

We now change over to the parametric plane of the complex variable

$$\zeta = \frac{\mu}{k} f'(z), \qquad z = \omega(\zeta) \tag{2.5}$$

By virtue of the boundary condition (2.4) and the previously mentioned assumption that there is a one-to-one relation between the contour L in the z-plane and the unit circle $\zeta = e^{i\gamma}$ in the ζ-plane, where $\gamma = -\theta$, and the elastic region is inside this circle (Fig. 1*b*).

The function $\omega(\zeta)$ may prove to be multi valued. When $z \to \infty$, let

$$f'(z) = f_0 + f_1/z + O(z^{-2}), \qquad f_0 = \mu^{-1}(\tau_{xz}^{\infty} - i\tau_{yz}^{\infty}), \qquad f_1 = F/2\pi\mu$$

where τ_{xz}^{∞}, τ_{yz}^{∞} are the stresses at an infinitely remote point, and F is the resultant vector of the forces applied to the boundary [6]. In order that the plastic region enclose the entire hole it is necessary that

$$f_0 = 0, \qquad f_1 \neq 0 \tag{2.6}$$

The first of these conditions follows from the principle of maximum modulus, and the second from the principle of correspondence of boundaries [7,8]; the conditions (2.6) are the conditions for the single-valuedness of the function $z = \omega(\zeta)$ when $|\zeta| \leqslant 1$. Moreover, the point $\zeta = 0$ is a first order pole since $\omega(\zeta) = f_1/\zeta + O(1)$ as $\zeta \to 0$. In general, the function ζ is not of one sheet. In particular, when $f_1 = 0$ the point $\zeta = f_0$ is always a branch point of the function $\omega(\zeta)$.

By (2.3), we obtain in the hodograph ζ-plane the following boundary value problem for the function $\omega(\zeta)$:

$$\operatorname{Re}[\zeta\omega(\zeta)] = \rho(\gamma) \qquad \text{for } \zeta = e^{i\gamma} \tag{2.7}$$

$$\rho(\gamma) = -\eta[t(-\gamma)]\sin\gamma + \xi[t(-\gamma)]\cos\gamma$$

where $\rho(\gamma)$ is a known continuous and single-valued function, and

$$\zeta\omega(\zeta) = f_1 + O(\zeta) \qquad \text{for } \zeta \to 0 \tag{2.8}$$

From the Schwarz formula [7,9,10] we obtain the solution of (2.7) and (2.8)

$$\omega(\zeta) = \frac{1}{2\pi\zeta}\int_0^{2\pi} \rho(\sigma)\frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}\,d\sigma \tag{2.9}$$

Taking the limit $\zeta \to e^{i\gamma}$, we obtain the parametric equation of the boundary L separating the elastic and plastic regions

$$z(\gamma) = e^{-i\gamma}\left[\rho(\gamma) - f_1 + \frac{1}{\pi}\int_0^{2\pi} \frac{\rho(\sigma)\, d\sigma}{1 - e^{i(\gamma-\sigma)}}\right] \tag{2.10}$$

The integral in Formula (2.10) is to be understood in the sense of the principal value. We note that after the determination of the boundary L it is necessary to check whether the assumptions made earlier are satisfied.

In order that the solution of the problem in the Form (2.9) and (2.10) should exist, it is necessary that the function $\rho(\gamma)$ should satisfy the two conditions

$$(1) \qquad |z(\gamma)| \geqslant \xi^2(t) + \eta^2(t) \quad \left(\arg z(\gamma) = \tan^{-1}\frac{\eta(t)}{\xi(t)}\right) \tag{2.11}$$

(2) the function $\arg z(\gamma)$ should be of one sign.

It should be noted that during the solution of the problem use has been made of the Prandtl diagram without an unloading section, which is the same as the indirect assumption that the work of the plastic deformation is positive everywhere in the plastic region. Thus the solution of the problem is valid only for those loading paths for which the successive elastic-plastic boundaries contain the preceding ones, or at least come into contact on some sections. Otherwise the shape of the boundary between the elastic and plastic regions, as well as the overall solution will depend on the path of loading. For loading paths satisfying the above conditions, the limitations (2.11) applied to the function $\rho(\gamma)$ will be sufficient for the existence of the solution of the initial elastic-plastic problem. The remark concerning the paths of loading as well as the condition (2.11) apply to all subsequent solutions of elastic-plastic problems.

3. The auxiliary boundary value problem. 1. Let it be required to determine a function $\omega(z)$, which is analytic in the whole half-plane $\operatorname{Im} z > 0$, from the nonlinear boundary conditions on the real axis

$$|\omega(t)| = \alpha(t) \quad (t \in L), \qquad \operatorname{Re}\left[(a(t) - ib(t))\,\omega(t)\right] = 0 \quad (t \in M) \tag{3.1}$$

where $a(t)$, $b(t)$, $\alpha(t)$ are almost everywhere continuous functions satisfying the Gel'der condition on the interval of continuity and at an infinitely remote point $(a + ib \neq 0)$; $L = L_1 + L_2 + \ldots + L_n$, where L_k are the intervals $-\infty < a_k \leqslant t \leqslant b_k < \infty$; and M is the manifold of points on the real axis lying outside L.

We will require at least integrability of the function $\omega(z)$ at the end points of the intervals $t = a_k$, $t = b_k$ as well as at the points of discontinuity of the coefficient $a - ib(t = c_k)$ and of the function $\alpha(t)(t = d_k)$.

We indicate the method of solution of the boundary value problem (3.1), which is based on the reduction of it to a nonlinear Riemann boundary-value problem, solvable in closed form with the aid of methods analogous to the classical method of solving linear boundary value problems developed in the monographs of Muskhelishvili [9] and Gakhov [10]. As regards the nonlinear boundary value problem and the related problems of nonlinear singular integral equations, they are the celebrated, chiefly qualitative investigations touching on the questions of existence and uniqueness of the solution. Some nonlinear problems, solved in closed form, were also treated in [11].

2. By canonical function of the nonlinear boundary value problem (3.1) will be denoted that sectionally holomorphic function $X(z)$ with a line of discontinuity on the real axis which is the canonical function of the Riemann problem

$$X^+(t) = G(t) X^-(t), \qquad G(t) = \begin{cases} -\dfrac{a(t) + ib(t)}{a(t) - ib(t)} & (t \in M) \\ 1 & (t \in L) \end{cases} \tag{3.2}$$

Moreover near the points $t = c_k$, the class of $X(z)$ coincides with the given class of the functions $\omega(z)$, and at the points a_k and b_k the function $X(z)$ is bounded.

The canonical function thus determined for the problem (3.1) can be written in the form [9,10]

$$X(z) = \prod_{k=1}^{n} (z - b_k)^{-\varkappa_k} e^{\Gamma(z)}, \qquad \Gamma(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ln G(\tau)}{\tau - z} d\tau \tag{3.3}$$

Let $\kappa = \kappa_1 + \ldots + \kappa_n$ be the index of the Riemann problem (3.2), which is determined in the usual manner ([10], p. 436).

The boundary condition (3.1) can be rewritten in the following way:

$$\omega(t) \overline{\omega(t)} = \alpha^2(t) \quad (t \in L), \quad (a - ib)\, \omega(t) + (a + ib) \overline{\omega(t)} = 0 \, (t \in M) \tag{3.4}$$

We introduce the function $\Phi(z)$, which is analytic in the whole z-plane except, perhaps, on the real axis:

$$\Phi(z)=\begin{cases}\omega(z)/X^{+}(z) & \text{for } \operatorname{Im} z>0\\ \bar{\omega}(z)/X^{-}(z) & \text{for } \operatorname{Im} z<0\end{cases} \tag{3.5}$$

With the aid of Formulas (3.5) and (3.2), the boundary condition (3.4) can be written in the form

$$\Phi^{+}\Phi^{-}=\alpha^{2}(t)\,X^{-2}(t)\quad (t\in L),\qquad \Phi^{+}-\Phi^{-}=0\quad (t\in M) \tag{3.6}$$

The function $\Phi(z)$ clearly has zero order at the points C_k and order $-\kappa$ at infinity.

Thus the problem (3.1) reduces to the nonlinear Riemann problem: to determine the function $\Phi(z)$ analytic outside the cut L under the condition

$$\Phi^{+}\Phi^{-}=\beta(t)\quad (t\in L),\qquad \beta(t)=\alpha^{2}(t)\,X^{-2}(t) \tag{3.7}$$

In [12] a treatment has been given of the nonlinear boundary value problem of the Riemann type

$$[\Phi^{+}(t)]^{n}=G(t)\Phi^{-}(t)+g(t)$$

where n is an integer >1, for a simple, smooth, closed contour which divides the plane into an interior and an exterior region. In this problem the index of the function $G(t)$ plays just as important a role as in the linear case. The nonlinear character of the problem is made evident by the fact that the constant obtained with the solution must satisfy the condition of absence of branch points in the solution. A particular solution of a problem of this type was considered in the earlier work [13]. We now note that when $n<0$ the index of $G(t)$ ceases to play an important role in questions of solubility, and what is determined is the inner nature of the function $\Phi(z)$ – the number of its zeros in the interior or exterior region. The problem (3.7) can be reduced to a linear Riemann boundary value problem for an analytic function, generally speaking, having a logarithmic singularity.

Note. The value of the function $G(t)$ on L has been taken equal to unity in (3.2) only for definiteness. In concrete problems it is more convenient to define $G(t)$ on L in a continuous manner so that the absolute value of the index κ will be as small as possible.

3. We consider the problem (3.7) for a simply connected region bounded by a simple smooth closed curve. We assume from the beginning that $\beta(t)$ satisfies the Gel'der condition and nowhere vanishes on the contour L,

which separates the interior D^+ and the exterior region D^-. The index of $\beta(t)$ is clearly equal to the difference between the number of zeros of the functions $\Phi^+(z)$ and $\Phi^-(z)$ in their regions of definition, D^+ and D^-.

We consider the two functions $\Phi^+(z)$ and $\Phi^-(z)$ defined by the expressions

$$\Phi^+(z) = M_1(z)\, e^{\Gamma^+(z)}, \quad \Phi^-(z) = \frac{M_2(z)}{M_1(z)} e^{-\Gamma^-(z)}, \quad \Gamma(z) = \frac{1}{2\pi i}\int_L \ln \frac{\beta(\tau)}{M_2(\tau)} \frac{d\tau}{\tau - z} \tag{3.8}$$

where $M_1(z)$ and $M_2(z)$ are arbitrary and, in general, nonanalytic functions which are extended continuously onto the contour L so that $M_1^+(t) = M_1^-(t)$, $M_2^+(t) = M_2^-(t)$. It is assumed that the integral on the contour L has a definite meaning, and for this the formula of Sokhotskii has been applied. The formulas (3.8) give a certain solution of the functional equation (3.7) for simply connected regions in the class of nonanalytic functions. Not studying the question of the degree of generality of this solution, we note, however, that, for some more general assumptions, it is easy to prove the uniqueness of the representation (3.8) in the class of analytic functions having isolated singularities. By narrowing the class of admissible functions $M_1(z)$ and $M_2(z)$ it is possible to determine all the required analytic solutions of the boundary value problem (3.7) for simply connected regions.

Let the analytic functions $\Phi^+(z)$ and $\Phi^-(z)$ have zero order everywhere in the regions of their definition, and let the index of the function $\beta(t)$ be equal to zero. Then in the general solution (3.8) it is obvious that it is necessary to take

$$M_1(z) = C, \qquad M_2(z) = 1 \tag{3.9}$$

where C is an arbitrary constant. If the value of the function $\Phi^-(z)$ is prescribed at infinity, the solution becomes unique.

Let the function $\Phi^+(z)$ have m zeros at the points $z = a_i$ of the region D^+, and let the function $\Phi^-(z)$ have n zeros at the points $z = b_i$ of the region D^-, whereby $m - n = \kappa$, where $\kappa = \operatorname{Ind} \beta(t)$. For definiteness, the coordinate origin will be taken to lie in the region D^+. Then the solution of the problem (3.7) can be determined up to an arbitrary multiplicative factor by means of Formula (3.8) in which it is clearly necessary to put

$$M_1(z) = C \prod_{i=1}^{m} (z - a_i) \prod_{i=1}^{n} (z - b_i)^{-1}, \qquad M_2(z) = z^{\kappa}\,. \tag{3.10}$$

If the functions $\Phi^+(z)$ and $\Phi^-(z)$ are not subject to the additional requirements concerning the number and location of the zeros, as was done here, then the boundary value problem (3.7) will have an infinite number of solutions determined by Formulas (3.10) for the arbitrary numbersɔers a_i, b_i, m, n, such that $a_i \in D^+$, $b_i \in D^-$, $m - n = \kappa$.

In this matter there is a most essential difference of the boundary value problem (3.7) from the linear Riemann boundary value problem [9,10].

Let the coefficient $\beta(t)$ have a zero or pole of integral order or let it have a finite number of discontinuities of the first kind. It is convenient to introduce the canonical function of the problem (3.7), which can be defined as the sectionally holomorphic function, which satisfies the condition (3.7) and has zero order everywhere in the finite part of the plane and order $-\kappa$ at infinity. The canonical function $X_0(z)$ of the problem (3.7) can be found from the formulas

$$X_0^+(z) = e^{\Gamma^+(z)}, \qquad X_0^-(z) = z^{\varkappa} e^{-\Gamma^-(z)} \tag{3.11}$$

where $\Gamma(z)$ can be determined from Formula (3.8) with $N_2(\tau) = \tau^{\kappa}$. τ^{κ}. With the aid of the $X_0(z)$, the cases when $\beta(t)$ has a zero or pole of integral order or has discontinuities of the first kind, can be studied in exactly the same way as that applied in the linear Riemann problem [9,10]. In particular, the discontinuities of the first kind can be removed by the introduction of auxiliary potential functions.

Note. The problem (3.7) for a multiply connected region bounded by a closed contour does not represent any essential features in comparison with the case of the simply connected region.

4. We consider the problem (3.7) for an open contour. To begin with, suppose that on an open curve L, consisting of n arcs with end points a_k and b_k, that the function $\beta(t)$ satisfies almost everywhere the Gel'der condition and that it does not vanish on a convergent sequence of points.

We consider the function $\Phi(z)$ defined by the expression

$$\Phi(z) = N(z)\, e^{\Gamma(z)}, \qquad \Gamma(z) = \frac{X_n(z)}{2\pi i}\left[\int_L \ln\frac{\beta(\tau)}{N^2(\tau)} \cdot \frac{d\tau}{X_n^+(\tau)(\tau - z)} + N_1(z)\right]$$
$$X_n(z) = \prod_{i=1}^{n} (z - a_i)^{1/2} (z - b_i)^{1/2} \tag{3.12}$$

Here the function $X_n(z)$ is analytic outside the cut L, and also $X_n(z) = Z^n + o(z^n)$ when $z \to \infty$; $N(z)$ and $N_1(z)$ are arbitrary nonanalytic functions, extended continuously onto the contour in such a way that $N_1^+(t) = N_1^-(t)$; $N^+(t) = N^-(t) = N(t)$; $N(t)$ satisfies the Gel'der condition almost everywhere on L. Besides, we will assume for simplicity that $\beta(\tau)/N^2(\tau)$ is bounded and does not vanish at the ends of the cuts, and that $N_1(z)$ cannot become infinite with order greater than 1/2.

Formulas (3.12) give the most general solution of the functional Equation (3.7) for an open contour L in the class of nonanalytic functions bounded at the end points of the cuts a_k and b_k.

It can be proved that, in general, the solution of the boundary value problem (3.7), at least in the class of analytic functions having only isolated singularities, will have, at the end points of the cuts a_k, b_k, either a nonintegrable essential singularity or will be bounded, so that from the requirement of integrability of the solution at the ends of the cuts a_k, b_k it follows that it is bounded in the neighborhood of these end points.

By restricting the class of admissible functions $N(z)$ and $N_1(z)$, it is possible to obtain from (3.12) all the required analytic solutions, which are bounded at the ends of the cuts, of the boundary value problem (3.7) for an open contour.

Let the analytic solution of the problem (3.7) have everywhere zero order. In this case, in Formula (3.12) it is necessary to put

$$N(z) = 1, \qquad N_1(z) = 0 \tag{3.13}$$

Moreover, the following $(n - 1)$ conditions must be satisfied

$$\int_L \frac{\tau^k \ln \beta(\tau)}{X_n^+(\tau)}\, d\tau = 0 \qquad (k = 0, 1, \ldots, n-2) \tag{3.14}$$

Let the analytic solution of the problem (3.7) have m zeros at the points c_i; $i = 1, \ldots, m$ (some or all the c_i may coincide). We will assume for definiteness that the point $z = 0$ lies on the contour L. In the general solution (3.12) for an open contour, it is clear that it is necessary to put

$$N(z) = z^{-m} \prod_{i=1}^{m} (z - c_i), \qquad N_1(z) = 0 \tag{3.15}$$

Moreover, the following $(n - 1)$ conditions must be satisfied

$$\int_L \frac{\tau^k}{X_n^+(\tau)} \ln \left[\beta(\tau)\, \tau^{2m} \left(\prod_{i=1}^{m} (\tau - c_i)^{2}\right)^{-1}\right] d\tau = 0 \qquad (\; = 0, 1, \ldots, n-2) \tag{3.16}$$

Making use of the formula which describes the behavior of a Cauchy integral in the neighborhood of the points on the contour, where its density has a logarithmic singularity ([10], p. 72), it is easy to find that the solution of the problem (3.7), given by Formulas (3.12), (3.15) and (3.16), is bounded in the vicinity of the point $z = 0$. Thus, in analogy with the case of the closed contour, for the complete determination of the problem it is necessary, in general, to prescribe the number and location of the zeros of the required solution; otherwise if they are not prescribed the number of solutions is infinite and the solutions are determined by Formulas (3.12), (3.15), (3.16) with arbitrary m, c_i. For a prescribed number of zeros m the solution can be determined with an accuracy up to m arbitrary constants.

Let the function $\beta(t)$ have a discontinuity of first order in the point $z = t_1$. We assume that the function $N_1(z)$ in the form (3.12) is bounded in the vicinity of the point $z = t_1$, and the function $N(z)$ is bounded and does not vanish in this point. Then when $z \to t_1$ by Formula (3.12)

$$\Phi(z) = (z - t_1)^{\frac{1}{2\pi i}\ln \chi} \cdot G_0(z), \quad \chi = \frac{\beta(t_1 - 0)}{\beta(t_1 + 0)} \qquad (3.17)$$

where $G_0(z)$ is a function bounded when $z = t_1$. We will choose the argument χ in relation to the given class of the solution in the point of discontinuity $z = t_1$.

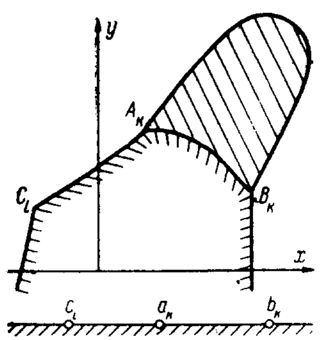

Fig. 2.

Note. Similarly, it is possible to derive the solution of the nonlinear boundary value problem of the type

$$[\Phi^+(t)]^n = G(t)\,\Phi^-(t) + g(t)$$

where n is an arbitrary integer, for a closed contour, and the solution of the problem

$$[\Phi^+(t)]^n = G(t)\,\Phi^-(t)$$

for an open contour, does not exist in general when $n = 0$. When $n > 0$ the basic characteristic of the problem is the index $G(t)$, and when $n < 0$ it is the number of zeros of the function $G(t)$. In the general case difficulties arise in ascertaining the character of the arbitrary complex constants which occur in the solution. In connection with the coefficients $G(t)$ and $g(t)$ of these problems as well as with the determination of the values of the function $\Phi(z)$ on the contour, it is possible to make a great amount of generalization analogous to the case of the linear problem (see [14, 15]).

4. The elastic-plastic problem for the exterior of a contour consisting of straight and curved lines in that case when the straight segments are free of loading and the segments of the curved arcs which are arbitrarily loaded are entirely within the plastic zone. 1. Let the curved boundary arc, which lies wholly in the plastic region (Fig. 2) be represented in the complex z-plane by the equations

$$x = \xi_k(t), \qquad y = \eta_k(t) \qquad (k = 1, \ldots, m)$$

where $\xi_k(t)$, $\eta_k(t)$ are continuous functions. To this same arc there has been applied a loading

$$\tau_{zn} = k\tau_k(t), \qquad |\tau_k(t)| \leqslant 1$$

Let A_k, B_k, C_i be the vertices of the polygon which forms the contour of the body; A_k, B_k are the points of the curved arcs some of which may extend to infinity, $i = 1, \ldots, n$. The equations of the straight lines free of loading have the form

$$y = x \tan \theta_j + d_j \qquad (j = 1, \ldots, m+n)$$

where θ_j is the angle between the jth line and the x-axis. The stresses in the plastic zones are given by Formulas (2.1) and (2.2), where $\xi(t)$, $\eta(t)$, $\tau(t)$ must be replaced by $\xi_k(t)$, $\eta_k(t)$, $\tau_k(t)$. The plastic region cannot extend to the straight lines perpendicular to the straight sections of the boundary and emanating from the vertices A_k, B_k. Otherwise there would be a curved arc on which $f'(z)$ = const, which is not possible.

On the unknown boundary between the elastic and plastic regions there is the condition on the continuity of the stresses (2.4), where θ is defined by Formula (2.1), and on the jth straight section of the boundary $y = x \tan \theta_j + d_j$ the condition for the absence of loading

$$\operatorname{Re}\,[(\tan \theta_j - i)\, f'(z)] = 0 \tag{4.1}$$

We pass to the parametric plane of the complex variable ζ with the aid of the conformal mapping $z = \omega(\zeta)$ such that the points C_i, A_k, B_k in the z-plane are transformed into the points c_i, a_k, b_k on the real axis in the ζ-plane and the elastic region is in the upper half-plane $\operatorname{Im} \zeta > 0$ (Fig. 2).

We introduce the notation

$$\mu f'[\omega(\zeta)] = kF(\zeta) \tag{4.2}$$

For the determination, From Formulas (4.1), (2.3), (2.4) and the equations of the lines, of the two functions $\omega(\zeta)$ and $F(\zeta)$ which are analytic in the upper half-plane $\mathrm{Im}\ \zeta > 0$, we obtain the boundary condition

$$|F(\zeta)| = 1 \quad \text{on } L, \qquad \mathrm{Re}\,[(\tan\theta_j - i)\,F(\zeta)] = 0 \quad \text{on } M \tag{4.3}$$

$$\mathrm{Re}\,[F(\zeta)\,\omega(\zeta)] = \rho(\theta) \quad \text{on } L, \qquad \mathrm{Re}\,[(\tan\theta_j + i)\,\omega(\zeta)] = -d_j \quad \text{on } M \tag{4.4}$$

Here L consists of the points of the real axis lying between a_k and b_k ($k = 1, 2, \ldots, m$), and M consists of the remaining points on the real axis; $\rho(\theta) = \eta[t(\theta)]\sin\theta + \xi_k[t(\theta)]\cos\theta$ on the segment (a_k, b_k); and the function $t(\theta)$ is defined by the relation $\theta = \alpha(t) - \beta(t)$ (see (2.1) to (2.2)).

The boundary value (4.3) is related to the type of problem treated in Section 3, since the function $F(\zeta)$ is determined independently of $\omega(\zeta)$. After finding the function $F(\zeta)$ and substituting it into the boundary condition (4.4) for the determination of $\omega(\zeta)$, we obtain a thoroughly studied Hilbert problem for the upper half-plane [9,10] (we note that $\theta = -\arg F(\zeta)$ on the basis of Formula (2.4)).

The elastic problem under the condition of antiplanar deformation is analogous to a two-dimensional problem in hydrodynamics [6]: in this the displacement w corresponds to the velocity potential, the stress vector τ corresponds to the velocity vector. The present case with the absence of dislocations corresponds in the hydrodynamic analogy to irrotational streamline flow. This analogy makes possible visual determination of the number and order of zeros of the function $F(\zeta)$, since those points where $F(\zeta)$ vanishes are critical points of the flow. For example, when the contour of the body is free of loading and there is a constant stress at infinity, the function $F(\zeta)$ has two zeros lying in M on the real axis. In general, their coordinates are not known beforehand and are to be determined from the solution of the problem (4.4), as well as the constants a_k, b_k, c_i (apart from three of these, which can be arbitrarily prescribed). The stated solution is valid also for the case when the body occupies the interior of a contour which consists of segments of straight lines and curves and when the straight segments are free of loading and the arbitrarily loaded segments of the curved arcs lie wholly within the plastic zone. For this it is essential to bear in mind the conditions, analogous to (2.11), which were imposed on the curved part of the boundary, and the character of the loading, as well as the note relating to the path of loading (Section 2).

It is curious, that the elastic problem cannot be expressed in quadratures, so that the elastic-plastic problem turns out, in principle, to be simpler than the elastic one. We point out that Galin treated the torsion of a bar of polygonal cross-section and the solution of the problem was reduced to the solution of a Fuchs-type differential equation [16]. Problems in the theory of torsion prove to be more complicated than those of antiplanar deformation because the right-hand side of the representation (1.7) contains the additional term $\mu \omega z$, where ω is the angle of twist per unit length of the bar.

2. In spite of the fact that, in principle, the solution of the above problem has been obtained in quadratures, the actual performance of the integrations meets with great difficulty. Therefore there is great interest in treating particular problems by simpler methods. A significant simplification is introduced if the hodograph plane (2.5) is used as the parametric ζ-plane, in those cases when the boundary of the elastic region in the z-plane is connected by a one-to-one relation with the known boundary in the hodograph plane. This occurs, for example, in the elastic-plastic problem for a half-plane with a notch, the walls of which are plane and the bottom arbitrarily loaded and completely contained in the plastic zone.

We consider in more detail the elastic-plastic problem for a half-plane with a straight crack l extending from the boundary of the half-plane. The surface of the crack and the boundary of the half-plane are free of stress, and a shear stress τ_∞ acts at infinity. On the plane $\zeta = (\mu/k)f'(z)$ the elastic region is mapped into the unit semi-circle with the cut by the line $\tau = \tau_\infty/k$. The resulting boundary value problem is easily solved with the aid of a double analytic continuation through the diameter of the circle and the arc of the circumference. The solution for all ζ, except $\zeta = 0$, is

$$z = l + \frac{2l\zeta\,(1-\zeta^2)}{\pi X\,(\zeta)} \int_0^\tau \frac{(t^2-1)\sqrt{(\tau^2-t^2)\,(\tau^{-2}-t^2)}}{(t^2+\zeta^2)\,(1+t^2\zeta^2)}\,dt \tag{4.5}$$

$$X\,(\zeta) = \sqrt{(\zeta^2+\tau^2)\,(\zeta^2+\tau^{-2})}$$

where $X(\zeta) = \zeta^2 + O(1)$ when $\zeta \to \infty$.

The integral in the form (4.5) can be expressed in terms of elliptic integrals of the first, second and third kind. However, the expressions obtained are unwieldy and not effective for numerical calculations. What is more effective is the asymptotic expansion of the function $z(\zeta)$ in terms of the dimensionless parameter $\tau = \tau_\infty/k$, which is assumed to be small.

We quote a part of the asymptotic series

$$z = l + \frac{l\zeta(1-\zeta^2)}{X(\zeta)}\left\{\frac{1}{\zeta(1-\zeta^2)}\left[\sqrt{1+\frac{\zeta^2}{\tau^2}}\left(1+\frac{1}{2}\tau^2\zeta^2 - \right.\right.\right.$$
$$\left. - \frac{1}{8}\tau^4\zeta^4 + \frac{5}{128}\tau^6\zeta^6\right) - \sqrt{1+\frac{1}{\tau^2\zeta^2}}\left.\left(1+\frac{\tau^2}{2\zeta^2} - \frac{\tau^4}{8\zeta^4} + \frac{5}{128}\frac{\tau^6}{\zeta^6}\right)\right] +$$
$$+ \frac{1}{\tau\zeta^2}\left[1 + \frac{\tau^2}{2}\left(1+\zeta^2+\frac{1}{\zeta^2}\right) + \frac{\tau^4}{8}\left(1-\zeta^4-\frac{1}{\zeta^4}-\zeta^2-\frac{1}{\zeta^2}\right) - \right.$$
$$\left.\left. - \frac{\tau^6}{16}\left(1+\zeta^2+\frac{1}{\zeta^2}\right)\right] + O(\tau^7)\right\} \tag{4.6}$$

Here $\sqrt{(1+z^2)} = z + O(z^{-1})$ as $z \to \infty$, and the expansion (4.6) is valid for all ζ, except $\zeta = 0$, when $\tau = \tau_\infty/k$ is small (practically up to values $\tau \sim 0.9$ for not too small or large ζ). In the limiting case $\tau_\infty = k$, Formula (4.5) gives

$$z = l + \frac{2l}{\pi}\left[\frac{\zeta^2-1}{\zeta(\zeta^2+1)} + \frac{1}{\zeta^2}\ \tan^{-1}\zeta - \tan^{-1}\frac{1}{\zeta}\right] \tag{4.7}$$

wherein $\tan^{-1} 0 = 0$.

We will now determine the boundary of the plastic region. Since $\zeta = e^{i\varphi}$ it is sufficient to use terms up to the fifth order in Formula (4.6). On the basis of (4.6) and (4.7) we obtain the equation of the contour of the plastic zone in the following form:

when $\tau \lessapprox 0.8$

$$\frac{x}{l} = 1 + \frac{\tau(1-\cos 2\varphi)}{2\sqrt{\cos 2\varphi + (1+\tau^4)/\tau^2}}\left[1 - \frac{\tau^2}{4}(1+2\cos\varphi) + O(\tau^4)\right] \tag{4.8}$$
$$\frac{y}{l} = \frac{\tau \sin 2\varphi}{2\sqrt{\cos 2\varphi + (1+\tau^4)/\tau^2}}\left[1 - \frac{\tau^2}{4}(1+2\cos\varphi) + O(\tau^4)\right]$$

when $\tau = 1$

$$\frac{x}{l} = 1 + \frac{2\sin\varphi}{\pi}\left(\tan\varphi - \frac{\pi}{2}\sin\varphi - \cos\varphi \ln\frac{\cos\varphi}{1+\sin\varphi}\right) \tag{4.9}$$
$$\frac{y}{l} = \frac{2}{\pi}\left(\sin\varphi - \frac{\pi}{4}\sin 2\varphi - \frac{1}{2}\cos 2\varphi \ln\frac{\cos\varphi}{1+\sin\varphi} + \frac{1}{2}\ln\frac{\cos\varphi}{1-\sin\varphi}\right)$$

We note the simple formula for the distance x^* of the point of intersection of the contour of the plastic region with the x-axis from the coordinate origin; which is obtained from (4.8) when $\varphi = \pi/2$

$$\frac{x}{l} = 1 + \tau^2\left[1 + \frac{3}{4}\tau^2 + O(\tau^4)\right] \tag{4.10}$$

Figure 3 shows the elastic-plastic boundaries, calculated with Formulas (4.9) and (4.10), for values of the loading parameter $\tau = \tau_\infty/k$ equal to 0.2, 0.5, 0.8, 1.0.

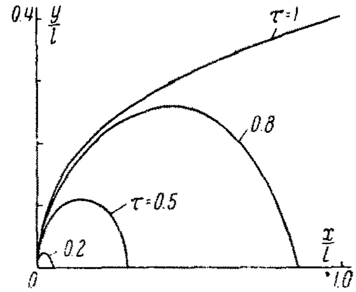

Fig. 3.

In the works [2,17] the solution of the elastic-plastic problem for a crack with small $\tau \ll 1$ was found and applied to the problem of the stability of a crack in shear.

3. We quote further the solution of the elastic-plastic problem for a body contained in the wedge $\theta_0 > \arg z > -\theta_0$, where $\pi \geqslant \theta_0 > 0$. On the sides of the wedge there is a constant prescribed displacement such that $w = h$ when $\arg z = \theta_0$ and $w = -h$ when $\arg z = -\theta_0$. The solution appears in the following way:

in the elastic region

$$f(z) = (h / \theta_0)\, i \ln z$$

in the plastic region

$$w = (h / \theta_0) \arg z, \qquad \tau = kie^{i \arg z} \tag{4.11}$$

The boundary between the elastic and plastic regions is $|z| = \mu h/\theta_0 k$.

In conclusion we note that the solution of the elastic-plastic problem for the boundary of a body consisting of straight lines and curves in that case when the straight segments are free of loading and the segments of the curves which are loaded lie wholly in the plastic zone can be carried over with inessential modifications to the solution of the analogous problem, if the existence of concentrated forces is allowed in the interior of the elastic region or on the straight-line boundaries of the body.

The author is indebted to L.Ia. Semenov and I.N. Balashov for assistance during the formulation of the problem.

BIBLIOGRAPHY

1. Trefftz, E., Über die Spannungsverteilung in tordierten Stäben bei teilweiser Überschreitung der Fliessgrenze. *ZAMM* Bd. 5, 1925.

2. Hult, J.A.H. and McClintock, F.A., Elastic-plastic stress and strain distributions around sharp notches under repeated shear. *Ninth Intern. Congr. for Appl. Mech.*, Vol. 8. Brussels, 1957.

3. Neuber, H., Theorie der Spannungskonzentration für beliebiges nichtlineares Spannungs-Dehnungs-Gesetz in Schubbeanspruchten Prismen. *Russk. per. Sb. Mekhanika* 4 (68). IL, 1961.

4. Kachanov, L.M., *Osnovy teorii plastichnosti (Foundations of the Theory of Plasticity)*. GITTL, 1956.

5. Sokolovskii, V.V., *Teoriia plastichnosti (Theory of Plasticity)*. GTTI, 1950.

6. Barenblatt, G.I. and Cherepanov, G.P., O khrupkikh treshchinakh prodol'nogo sdviga (On brittle cracks in longitudinal shear). *PMM* Vol. 25, No. 6, 1961.

7. Lavrent'ev, M.A. and Shabat, B.V., *Metody teorii funktsii kompleksnogo peremennogo (Methods of the Theory of Functions of a Complex Variable)*. Fizmatgiz, 1958.

8. Goluzin, G.M., *Geometricheskaia teoriia funktsii kompleksnogo peremennogo (Geometric Theory of Functions of a Complex Variable)*. Gostekhizdat, 1952.

9. Muskhelishvili, N.I., *Singuliarnye integral'nye uravneniia (Singular Integral Equations)*. GTTI, 1946.

10. Gakhov, F.D., *Kraevye zadachi (Boundary Value Problems)*. Fizmatgiz, 1958.

11. Natalevich, V.K., Nelineinye singuliarnye integral'nye uravneniia i nelineinye kraevye zadachi teorii analiticheskikh funktsii (Nonlinear singular integral equations and nonlinear boundary value problems of the theory of analytic functions). *Uch. zap. Kazanskogo un-ta* Vol. 112, No. 10, 1952.

12. Arzhanov, G.V., O nelineinoi kraevoi zadache tipa zadachi Rimana (On the nonlinear Riemann boundary value problem). *Sib. mat. zhurnal* Vol. 2, No. 4, 1961.

13. Solov'ev, P.V., Ob odnoi granichnoi zadache v teorii analiticheskikh funktsii (On a boundary value problem in the theory of analytic functions). *Dokl. Akad. Nauk SSSR* Vol. 33, No. 3, 1941.

14. Rogozhin, V.S., Kraevye zadachi Rimana i Gil'berta v klasse obobshchennykh funktsii (Riemann and Hilbert boundary value problems in the class of generalized functions). *Sib. mat. zhurnal* Vol. 2, No. 5, 1961.

15. Daniliuk, I.I., O zadache Gil'berta s izmerimymi koeffitsientami (On the Hilbert problem with measurable coefficients). *Sib. mat. zhurnal* Vol. 1, No. 2, 1960.

16. Galin, L.A., Uprugo-plasticheskoe kruchenie prizmaticheskikh sterzhnei poligonal'nogo secheniia (Elastic-plastic torsion of prismatic bars with polygonal cross-section). *PMM* Vol. 8, No. 4, 1944.

17. McClintock, F.A., Ductile fracture instability in shear. *J. Appl. Mech.* 25, No. 4, 1958.

Translated by D.McV.